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Transfer Matrix Spectrum for Lattice Classical O(N) Ferromagnetic Spin Systems at High Temperature

Ricardo S. Schor[1] **and Michael O'Carroll**[2]

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We obtain new properties of d-dimensional lattice ferromagnetic classical N-component vector spin systems in the high temperature region. Each model is characterized by a single site a priori single spin probability distribution (sspd) which we take to be rotationally invariant. Associated with the model is a discrete imaginary time lattice quantum field theory which is known to contain particles of mass m. Letting $\langle \cdot \rangle$ denote the sspd expectation we show that there exists two bound states below but near the two-particle threshold $2m$ if, with $\vec{s} = (s_1, s_2,..., s_N)$, $\alpha_N \equiv \langle (\vec{s} \cdot \vec{s})^2 \rangle - \frac{N+2}{N} \langle \vec{s}^2 \rangle^2 > 0$; if $\alpha_N < 0$ there are no bound states. These results are obtained using a lattice version of the Bethe–Salpeter equation in a ladder approximation.

KEY WORDS: Transfer matrix spectrum; classical ferromagnetic O(N) spin systems, decay of correlations; bound states; high-temperature ferromagnetic spin systems; Gaussian domination inequalities.

In this paper we obtain new properties of d-dimension lattice ferromagnetic classical vector spin systems in the high temperature region ($\beta \ll 1$). Each such system is characterized by a single site "a priori" spin probability distribution. Associated with the correlation functions (cf's) of a system is an imaginary time lattice quantum field theory with Hamiltonian energy and momentum operators living on a $d-1$ dimensional sublattice. The Hamiltonian is minus the logarithm of the transfer matrix.[1,2] The new properties are found by a detailed study of the interaction of the particles of this underlying field theory.

[1] Departamento de Física ICEx, Universidade Federal de Minas Gerais, Belo Horizonte, Minas Gerais, Brazil.

[2] Departamento de Matemática, ICMC-USP, C.P. 668, 13560–970 São Carlos, São Paulo, Brazil; e-mail: ocarroll@icmc.sc.usp.br

In the past much attention has been devoted to showing various cf inequalities for these systems[1] in the case of scalar or two-component (Abelian) spins. These inequalities, are used, for example to determine bounds on critical temperature, bounds on critical exponents, exponential decay above the critical point and triviality of some $d \geqslant 4$ quantum field theory models.[1]

In particular for a subclass of these models Gaussian domination inequalities are shown. It is not known whether or not the equalities hold for the case of three or more component spin systems. For Gaussian domination inequalities in the case of a scalar spin ($N = 1$), where $\langle s^m \rangle$ denotes the mth moment of the a priori single spin distribution, the ssd moment expectations are required to obey the collection of inequalities

$$\langle s^{2k} \rangle \leqslant (2k-1)(2k-3) \cdots (3)(1) \langle s^2 \rangle, \qquad k \geqslant 2.$$

For a Gaussian ssd equality holds in the above. For these models the 4-pt. cf decay rate is greater than or equal to twice the 2-pt. cf decay rate (see ref. 1). In the case of the 4pt. function the decay rate is that associated with the truncated 4pt. function with the points grouped into two pairs, each pair taken at coinciding points.

Recently in refs. 3–5 we have determined the low-lying e-m spectrum of a complementary class of models. In these models there is a richer particle spectrum and two-particle bound states occur which imply, for example, that the 4-pt. cf decay rate is less than twice that of the 2-pt. cf. Here we extend our determination of the e-m spectrum up to the two-particle threshold to vector spin models with rotationally invariant ssd. Due to the complexity of the interaction of the components of the spin we find a rich 2-particle bound state spectrum for a class of models with ssd restrictions; if the restrictions do not hold no bounds states are present. The ssd distribution restrictions that we find are generalizations to the case of vector spins of the scalar and abelian spin ssd restrictions occurring in the case of Gaussian domination. The new results on the spectrum and spectral multiplicities are made precise below and go beyond the spectral results obtained in refs. 4 and 5.

The partition function of the model we consider is given by the formal expression

$$Z = \int \exp\left[\beta \sum \vec{s}(\mathrm{x}) \cdot \vec{s}(y) \right] \prod_z e^{-V(|\vec{s}(z)|)} \, d\vec{s}(z)$$

where $\mathrm{x} = (\mathrm{x}_i, \vec{\mathrm{x}}) \in Z^d$, $\vec{s}(x) = (s_1(\mathrm{x}), s_2(\mathrm{x}), \ldots, s_N(\mathrm{x})) \in R^N$ and the sum is over unordered nearest neighbor pairs. $e^{-V(|\vec{s}(z)|)} \, d\vec{s}(x)$ is referred to as the single spin distribution and is taken to be rotationally invariant and even. Also $V(u)$ is bounded below and grows at infinity at least quadratically.

For small $\beta > 0$ infinite lattice correlation functions (cf), denoted by $\langle \cdot \rangle$, are obtained by a polymer expansion, are translationally invariant, analytic in β for small $|\beta|$, and truncated (connected) cf's have exponential free decay.[1,6]

An associated imaginary discrete time lattice quantum field theory is constructed using standard methods furnishing a Hilbert space $\mathscr{H}$, commuting self-adjoint energy-momentum (e-m) operators $H \geqslant 0$, P_{k}, $\mathrm{k} = 1, 2, \ldots, d-1$, time zero field operators $\hat{s}_i(\vec{\mathrm{x}}) = e^{i\vec{P}\cdot\vec{\mathrm{x}}}\hat{s}_i e^{-i\vec{P}\cdot\vec{\mathrm{x}}}$ and vacuum vector Ω. Vacuum expectation values of products of imaginary-time Heisenberg operators are related to cf by the Feynman–Kac (F-K) formula, for $\mathrm{x}_i(t_i, \vec{\mathrm{x}}_i)$ and $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$,

$$\begin{aligned}(\Omega, \hat{s}_{i_1} e^{-H(t_2-t_1)} e^{i\vec{P}\cdot(\vec{\mathrm{x}}_2-\vec{\mathrm{x}}_1)} \hat{s}_{i_2} \cdots e^{-H(t_n-t_{n-1})} e^{-\vec{P}\cdot(\vec{\mathrm{x}}_{\mathrm{n}}-\vec{\mathrm{x}}_{\mathrm{n}-1})} \hat{s}_{i_n} \Omega) \\ = \langle s_{i_1}(\mathrm{x}_1) \cdots s_{i_n}(\mathrm{x}_n) \rangle.\end{aligned}$$

Decay rates of cf's are related to the e-m spectrum, of the associated quantum field theory and the low-lying e-m spectrum has a particle interpretation. We state what is known about the spectrum generated by the one-particle states $\hat{s}_i(\vec{\mathrm{x}})\,\Omega$ and the two-particle states $\hat{s}_i(\vec{\mathrm{x}})\,\hat{s}_j(\vec{\mathrm{y}})\,\Omega$. E-m spectral points are denoted by $(E, \vec{p})$, $E \geqslant 0$, $\vec{p} \in T_{d-1}$ (the $d-1$ dimensional torus) and the point $(E, \vec{0})$ is referred to as a mass.

The e-m spectrum associated with the one-particle states consists of an isolated dispersion curve $w(\vec{p}) \geqslant w(\vec{0}) \equiv m$ and $m(\beta) = -\ln \frac{N}{\beta\langle \vec{s}^{\,2}\rangle} - \frac{2\beta\langle \vec{s}^{\,2}\rangle(d-1)}{N} + 0(\beta^2)$. Here we use $\langle s_i^k \rangle$ to denote the ssd expectation. $2m$ is called the two-particle threshold. For rotationally invariant states $\hat{s}(\vec{\mathrm{x}})\cdot\hat{s}(\vec{\mathrm{y}})\,\Omega$ it is shown in refs. 3–5 that there is a bound state with mass $m_b < 2m$, but near $2m$, given by $m_b = 2m - \ln[\frac{N(\langle(\vec{s}\cdot\vec{s})^2\rangle - \langle \vec{s}^{\,2}\rangle^2)}{2\langle \vec{s}^{\,2}\rangle^2}] + 0(\beta)$ if the ssd has the property that $\alpha_N \equiv \langle (s\cdot s)^2\rangle - \frac{N+2}{N}\langle s\cdot s\rangle^2 > 0$.

If $\alpha_N < 0$ there is no bound state. For a Gaussian ssd $\alpha_N = 0$ and there is no mass spectrum in $(m, 2m)$.

We now turn to the determination of all the mass spectrum in $(m, 2m)$. We recall from refs. 3–5 that a mass spectral point is detected as a $\mathrm{Im}\,\mathrm{k}_o > 0$ singularity in

$$\begin{aligned}&(\tilde{f}, \tilde{D}(\mathrm{k}_o)\,\tilde{f}) \\ &\quad \equiv \int dp^{d-1}\,dq^{d-1} \tilde{f}_{i_2 i_1}(\vec{p})\,\tilde{D}_{i_1 i_2 i_3 i_4}(\vec{p}, \vec{q}, \mathrm{k}_0)\,\tilde{f}_{i_3 i_4}(\vec{q}) \\ &\quad = \int_0^\infty \int_{T_{d-1}} \frac{\sinh E}{\cosh E - \cos \mathrm{k}_o}\,(2\pi)^{3(d-1)}\,\delta(\vec{q})\cdot d(\theta(f), \mathscr{E}(E, \vec{q})\,\theta(f))\end{aligned}$$

where $\mathscr{E}(E, \vec{q})$ is the spectral family associated with H, $\vec{P}$. $\theta(f) = \sum_{i,j,\vec{x}} f_{ij}(\vec{x})$ $\theta_{ij}(-\vec{x})$, $\vec{x} \in Z^{d-1}$, where $\theta_{ij}(\vec{\eta}) = \hat{s}_i(\vec{0})\ \hat{s}_j(\vec{\eta})\ \Omega - (\Omega, \hat{s}_i(\vec{0})\ \hat{s}_j(\vec{\eta})\ \Omega)\ \Omega$. $\tilde{D}(\vec{p}, \vec{q}, k)$ is the Fourier transform of the partially truncated 4-point cf

$$D_{i_1 i_2 i_3 i_4}(x_1 x_2 x_3 x_4) \equiv \langle s_{i_1}(x_1)\ s_{i_2}(x_2)\ s_{i_3}(x_3)\ s_{i_4}(x_4)\rangle - \langle s_{i_1}(x_1)\ s_{i_2}(x_2)\rangle\langle s_{i_3}(x_3)\ s_{i_4}(x_4)\rangle$$

expressed in the relative coordinates $\vec{\xi} = \vec{x}_2 - \vec{x}_1$, $\vec{\eta} = \vec{x}_4 - \vec{x}_3$, $\tau = x_3 - x_2$ with momentum variables $\vec{p}$, $\vec{q}$, k respectively. By abuse of notation we denote it by $D(\vec{\varepsilon}, \vec{\eta}, \tau)$, f is a function of the space variables only and $\tilde{f}$ denotes its Fourier transform. The k_0 in $\tilde{D}(k_0)$ means that we take $k = (k_0, \vec{k} = 0)$.

To analyze D we use a Bethe–Salpeter (B-S) equation for it which is analogous to the resolvent equation associated with the non-relativistic two-body quantum mechanical Hamiltonian where D is the analog of the interacting resolvent and

$$D^0_{i_1 i_2 i_3 i_4}(x_1 x_2 x_3 x_4) \equiv \langle s_{i_1}(x_1)\ s_{i_3}(x_3)\rangle\langle s_{i_2}(x_2)\ s_{i_4}(x_4)\rangle + \langle s_{i_1}(x_1)\ s_{i_4}(x_4)\rangle\langle s_{i_2}(x_2)\ s_{i_3}(x_3)\rangle$$

is the analog of the free Hamiltonian resolvent. Here the variables are restricted by $x_1^0 = x_2^0$ and $x_3^0 = x_4^0$, which we call equal times, and is different than the case of Euclidean quantum field theory where there is no such restriction (see refs. 7 and 8).

To obtain the B-S kernel of the B-S equation, the kernel variables restricted to $x_1^0 = x_2^0$ and $x_3^0 = x_4^0$, $D = D^0 + DKD^0$ we find, by expanding the cf's in β using $\langle s_i(x)\ s_j(y)\rangle = s_{ij}\langle s_1^2\rangle\ \delta(x, y) + 0(\beta)$,

$$D^0_{i_2 i_2 i_3 i_4}(x_1 x_2 x_3 x_4) = \left[\frac{\vec{s}^2}{N}\right]^2 [\delta_{i_1 i_3}\ \delta_{i_2 i_4}\ \delta(x_1 x_3)\ \delta(x_2 x_4) + \delta_{i_1 i_4}\ \delta_{i_2 i_3}\ \delta(x_1 x_4)\ \delta(x_2 x_2)],$$

$$\begin{aligned} D_{i_1 i_2 i_3 i_4}(x_1 x_2 x_3 x_4) &= [\delta_{i_1 i_3}\ \delta_{i_2 i_4}\ \delta(x_1 x_3)\ \delta(x_2 x_4) \\ &\quad + \delta_{i_1 i_4}\ \delta_{i_2 i_3}\ \delta(x_1 x_4)\ \delta(x_2 x_3)]\langle s_1^2\rangle^2 + \delta(x_1 x_2)\ \delta(x_2 x_3)\ \delta(x_3 x_4) \\ &\quad \cdot \{(\delta_{i_1 i_2}\ \delta_{i_3 i_4} + \delta_{i_1 i_3}\ \delta_{i_2 i_4} + \delta_{i_1 i_4}\ \delta_{i_2 i_3}) \cdot [\langle s_1^2 s_2^2\rangle - \langle s_1^2\rangle^2] + \delta_{i_1 i_2}\ \delta_{i_2 i_3}\ \delta_{i_3 i_4} \\ &\quad \cdot [\langle s_1^4\rangle - 3\langle s_1^2 s_2^2\rangle]\} + 0(\beta) \end{aligned}$$

and acting on functions $f_{ij}(\mathrm{x}, y)$ with $\mathrm{x}^0 = y^0$ and $f_{ij}(\mathrm{x}, y) = f_{ij}(y, \mathrm{x})$ we have

$$(D^0 f)_{i_1 i_2}(\mathrm{x}_1 \mathrm{x}_2) = 2\left[\frac{\langle \vec{s}^{\,2}\rangle}{N}\right]^2 f_{i_1 i_2}(\mathrm{x}_1 \mathrm{x}_2) + 0(\beta),$$

$$\begin{aligned}(Df)_{i_1 i_2}(\mathrm{x}_1 \mathrm{x}_2) = 2\langle s_1^2\rangle^2 f_{i_1 i_2}(\mathrm{x}_1 \mathrm{x}_2) + \delta(\mathrm{x}_1 \mathrm{x}_2)\Bigg\{\Bigg[&\delta_{i_1 i_2} \sum_{\mathrm{k}} f_{\mathrm{kk}}(\mathrm{x}, \mathrm{x}) \\ &+ 2 f_{\mathrm{ij}}(\mathrm{x}, \mathrm{x})\Big] \cdot [\langle s_1^2 s_2^2\rangle - \langle s_1^2\rangle^2] \\ &+ \delta_{ij} f_{ii}(\mathrm{x}, \mathrm{x})[\langle s_1^4\rangle - 3\langle s_1^2 s_2^2\rangle\Bigg]\Bigg\} + 0(\beta).\end{aligned}$$

Thus

$$(D_0^{-1} f)_{ij}(\mathrm{x}, y) = \frac{1}{2}\left(\frac{N}{\langle \vec{s}^{\,2}\rangle}\right)^2 f_{ij}(\mathrm{x}, y),$$

$$\begin{aligned}(D^{-1} f)_{ij}(\mathrm{x}, y) = (1 - \delta(\mathrm{x}, y))\,&\frac{1}{2\langle s_1^2\rangle^2} f_{ij}(\mathrm{x}, y) + 0(\beta) \\ &+ (1 - \delta_{ij})\frac{1}{2\langle s_1^2 s_2^2\rangle} f_{ij}(\mathrm{x}, \mathrm{x})\,\delta(\mathrm{x}, \mathrm{y}) \\ &+ \delta_{ij}\,\delta(\mathrm{x}, \mathrm{y})\frac{1}{[\langle \mathrm{s}_1^4\rangle - \langle \mathrm{s}_1^2 \mathrm{s}_2^2\rangle]} f_{ii}(\mathrm{x}, \mathrm{x}) \\ &- \gamma_3\,\delta_{ij}\,\delta(\mathrm{x}, y)\sum_{\mathrm{k}} f_{\mathrm{kk}}(\mathrm{x}, \mathrm{x}),\end{aligned}$$

and thus $K = D_0^{-1} - D^{-1}$ acting on f is

$$\begin{aligned}(Kf)_{ij}(\mathrm{x}, y) = (1 - \delta(\mathrm{x} y))\,0(\beta) &+ (1 - \delta_{ij})\,\delta(\mathrm{x}, y)\,\gamma_1 f_{ij}(\mathrm{x}, \mathrm{x}) \\ &+ \delta_{ij}\gamma_5\,\delta(\mathrm{xy})\,f_{ii}(\mathrm{x}, \mathrm{x}) + \delta_{ij}\gamma_3 \sum_{\mathrm{k}} f_{\mathrm{kk}}(\mathrm{x}, \mathrm{x}).\end{aligned}$$

In the above

$$\gamma_1 = \frac{1}{2}\left[\frac{1}{\langle s_1^2\rangle^2} - \frac{1}{\langle s_1^2 s_2^2\rangle}\right], \qquad \gamma_5 = \frac{1}{2\langle s_1^2\rangle^2} - \frac{1}{[\langle s_1^4\rangle - \langle s_1^2 s_2^2\rangle]}$$

and

$$\gamma_3 = \frac{\langle s_1^2 s_2^2\rangle - \langle s_1^2\rangle^2}{[\langle s_1^4\rangle - \langle s_1^2 s_2^2\rangle][\langle s_1^4\rangle + (N-1)\langle s_1^2 s_2^2\rangle - N\langle s_1^2\rangle^2]}.$$

In terms of the spin zero and spin two tensor projections

$$P^0_{i_1 i_2 i_3 i_4} = \delta_{i_1 i_2} \frac{1}{N} \sum_{\mathrm{k}} \delta_{\mathrm{k}i_3}\, \delta_{\mathrm{k}i_4},$$

$$P^2_{i_1 i_2 i_3 i_4} = \frac{1}{2}\left[\delta_{i_1 i_3}\, \delta_{i_2 i_4} + \delta_{i_1 i_4}\, \delta_{i_2 i_3} - \frac{2}{N} \delta_{i_1 i_2} \sum_{\mathrm{k}} \delta_{\mathrm{k}i_3}\, \delta_{\mathrm{k}i_4} \right]$$

we can write

$$(KP^2 f)_{ij}\,(\mathrm{x}, \mathrm{x}) = \gamma_1 (P^2 f)_{ij}\,(\mathrm{x}, \mathrm{x}) + 0(\beta)$$
$$(KP^0 f)_{ij}\,(\mathrm{x}, \mathrm{x}) = [\gamma_2 + (N-1)\,\gamma_3](P^0 f)_{ij}\,(\mathrm{x}, \mathrm{x}) + 0(\beta)$$

where $\gamma_2 = \gamma_5 + \gamma_3$. Thus γ_1 and $\gamma_2 + (N-1)\,\gamma_3$ are the eigenvalues of K to $0(\beta)$ and they have the rotationally invariant forms

$$\gamma_1 = \frac{N^2}{2\langle \vec{s}^{\,2} \rangle^2} \left[\frac{\langle (\vec{s}\cdot\vec{s}) \rangle^2 - \frac{N+2}{N} \langle \vec{s}\cdot\vec{s} \rangle^2}{\langle (\vec{s}\cdot\vec{s})^2 \rangle} \right] \equiv \lambda_2$$

$$\gamma_2 + (N-1)\,\gamma_3 = \frac{N^2}{2\langle \vec{s}^{\,2} \rangle^2} \left[\frac{\langle (\vec{s}\cdot\vec{s}) \rangle^2 - \frac{2+N}{N} \langle \vec{s}^{\,2} \rangle^2}{\langle (\vec{s}\cdot\vec{s})^2 \rangle - \langle \vec{s}^{\,2} \rangle^2} \right] \equiv \lambda_0$$

where we use the relations

$$\langle s_1^2 s_2^2 \rangle = \frac{\langle (\vec{s}\cdot\vec{s})^2 \rangle}{N(N+2)} = \frac{1}{3} \langle s_1^4 \rangle, \; \langle (\vec{s}\cdot\vec{s})^2 \rangle = N \langle s_1^4 \rangle + N(N-1) \langle s_1^2 s_2^2 \rangle.$$

These formulas are obtained by calculating the multi-dimensional Gaussian integral $\int s_1^4 e^{-\alpha |\vec{s}|^2}\, d^N s$ in two ways: using Cartesian coordinates and in spherical coordinates.

We call L, the β-independent approximation to K, the ladder approximation, i.e., $L(\vec{\xi}, \vec{\eta}, \tau) = \delta(\vec{\xi})\, \delta(\vec{\eta})\, \delta(\tau)\, \ell$. It is local in the space-time coordinates so that the B-S equation in relative coordinates and Fournier transformed in the τ variable only becomes

$$\hat{D}(\vec{\xi}, \vec{\eta}) = \hat{D}^0(\vec{\xi}, \vec{\eta}) + \hat{D}(\vec{\xi}, \vec{0})\, \ell \hat{D}^0(\vec{0}, \vec{\eta})$$

where we have suppressed the $\mathrm{k} = (\mathrm{k}_0, \vec{\mathrm{k}} = 0)$ dependence. ℓ is the matrix $\ell = \lambda_0 P^0 + \lambda_2 P^2$. Setting $\vec{\eta} = 0$ and solving for $\hat{D}(\vec{\xi}, \vec{0})$ we find

$$\hat{D}(\vec{\xi}, \vec{\eta}) = \hat{D}^0(\vec{\xi}, \vec{\eta}) + \hat{D}^0(\vec{\xi}, \vec{0})(1 - \ell\, \hat{D}^0(0, 0))^{-1}\, \ell\, \hat{D}^0(\vec{0}, \vec{\eta})$$

where

$$\hat{D}^0_{i_1 i_2 i_3 i_4}(\vec{0}, \vec{0}) = \int [S_{i_1 i_3}(\tau)\, S_{i_2 i_4}(\tau) + S_{i_1 i_4}(\tau)\, S_{i_2 i_3}(\tau)] \cdot e^{-i\mathrm{k}_0 \tau_0}\, d\tau^0\, d\tau$$

$$= (\delta_{i_1 i_3}\, \delta_{i_2 i_4} + \delta_{i_1 i_4}\, \delta_{i_2 i_3})\, R = Qr$$

and $r = \int \langle s_1(0)\, s_1(\tau)\rangle^2\, e^{-i\mathrm{k}_0 \tau_0}\, d\tau_0\, d\vec{\tau}$. Q commutes with P_0 and P_2 so that

$$\ell \vec{D}_0(\vec{0}, \vec{0}) = (\lambda_0 P^0 + \lambda_2 + P^2)\, Qr$$

has eigenvalues $2\lambda_0 r$ and $2\lambda_2 r_1$ of multiplicity one and $\frac{N(N-1)}{2}$, respectively.

The k_0 singularities of $(f, \tilde{D} f)$ in $\operatorname{Im} \mathrm{k}_0 \in (0, 2m)$ come from the solutions of, with $\mathrm{k}_0 = i(2m - \varepsilon)$,

$$2\lambda_i \sum_{\tau_0, \vec{\tau}} \langle s_1(0)\, s_1(\tau)\rangle^2 e^{2m\tau_0} e^{-\varepsilon\tau_0} = 1.$$

We give an intuitive argument for the bound state formula based on the behavior of $\langle s_1(0)\, s_1(\tau)\rangle$. A rigorous argument using the convolution form in momentum space of the above condition and the spectral representation of the two-point function can be given.

Expanding in β to leading order we have $\langle s_1(0)\, s_1(\tau)\rangle \approx \beta^{|\tau_0| + |\vec{\tau}|} \langle s_1^2\rangle^{|\tau_0| + |\vec{\tau}| + 1}$.

This leading behavior follows by expanding the numerator in the two-point function. There must be a chain of overlapping bonds connecting the two-points (otherwise the integral over spins is zero) and upon taking a chain of minimal length the above term results. To rigorously control all contributions more sophisticated methods, i.e., the polymer expansion, are used. Also

$$m = \ln\left(\frac{N}{\beta\langle \vec{s}^{\,2}\rangle}\right) - 2\beta\left(\frac{\langle \vec{s}^{\,2}\rangle}{N}\right)(d-1) + 0(\beta^2)$$

so that the bound state condition becomes, to leading order in β,

$$2\langle s_1^2\rangle^2\, \lambda_{\mathrm{k}} (1 - e^{-\varepsilon_{\mathrm{k}}})^{-1} = 1, \qquad \mathrm{k} = 0, 2,$$

or

$$e^{-\varepsilon_0} = \frac{2\langle \vec{s}^{\,2}\rangle^2}{N(\langle (\vec{s}\cdot\vec{s})^2\rangle - \langle \vec{s}^{\,2}\rangle^2)},$$

and

$$e^{-\varepsilon_2} = \frac{N+2}{N} \frac{\langle \vec{s}^2 \rangle^2}{\langle (\vec{s} \cdot \vec{s})^2 \rangle}.$$

In both cases the condition for the existence of a bound state, i.e., $e^{-\varepsilon_k} < 1$, $k = 0, 2$ is

$$\langle (\vec{s} \cdot \vec{s})^2 \rangle > \frac{N+2}{N} \langle \vec{s}^2 \rangle^2.$$

The masses are given by $-\ln \varepsilon_k$ and we now show that the scalar bound state mass $2m(\beta) - \delta_0$ is smaller than the spin two bound state mass $2m(\beta) - \delta_2$. This will be true if $\delta_2 < \delta_0$ or

$$\ln \frac{2\langle (\vec{s} \cdot \vec{s})^2 \rangle}{(N+2)(\langle (\vec{s} \cdot \vec{s})^2 \rangle - \langle \vec{s}^2 \rangle^2)} < 0$$

which holds if

$$\langle (\vec{s} \cdot \vec{s})^2 \rangle > \frac{N+2}{N} \langle \vec{s}^2 \rangle^2$$

which is precisely the condition imposed for the existence of the two bound states.

We now show how to rigorously obtain the ladder approximation equation for bound states. Using the spectral representation for the two-point functions[4, 5] occurring in D^0 we find that the action of $\tilde{D}^0(\mathrm{k}_0)$ on functions with the property $f_{ij}(\vec{p}) = f_{ij}(-\vec{p})$ is given by

$$(\tilde{D}^0(\mathrm{k}_0)\, f)_{ij}\, (\vec{p}) = 2(2\pi)^{3(d-1)} \int_0^\infty \int_0^\infty \frac{\sinh(E+E')}{\cosh(E+E') - \cos \mathrm{k}^0} \cdot d\sigma_{\vec{p}}(E)\, d\sigma_{\vec{p}}(E')\, f_{ij}(\vec{p}),$$

i.e., multiplication by the function

$$H(\vec{p}, \mathrm{k}_0) = 2(2\pi)^{3(d-1)} \int_0^\infty \int_0^\infty \frac{\sinh(E+E')}{\cosh(E+E') - \cos \mathrm{k}^0}\, d\sigma_{\vec{p}}(E)\, d\sigma_{\vec{p}}(E').$$

The B-S equation in momentum space is given by

$$\tilde{D}(\vec{p}, \vec{q}, \mathrm{k}^0) = \tilde{D}^0(\vec{p}, \vec{q}, \mathrm{k}^0) + (2\pi)^{-2(d-1)} \int \tilde{D}(\vec{p}, \vec{p}\,', \mathrm{k}^0) \cdot \tilde{K}(\vec{p}\,', \vec{q}\,', \mathrm{k}^0)\, \tilde{D}^0(\vec{q}\,', \vec{q}, \mathrm{k}^0)\, d\vec{p}\,'\, d\vec{q}\,'$$

and suppressing the k^0 dependence, we write

$$\begin{aligned}\tilde{D} &= \tilde{D}^0 + (2\pi)^{-2(d-1)}\,\tilde{D}\tilde{K}\tilde{D}^0\\ &= \tilde{D}^0(1-(2\pi)^{-2(d-1)}\,\tilde{K}\tilde{D}^0)^{-1}.\end{aligned}$$

From the analysis of the two-pt. function

$$d\sigma_{\vec{p}}(E) = Z(\vec{p},\beta),\ \delta(E-w(\vec{p}))\,dE + d\hat{\sigma}_{\vec{p}}(E)$$

where $Z(\vec{p},\beta) = (2\pi)^{-(d-1)}\frac{\langle\vec{s}^2\rangle}{N}(1+0(\beta))$ and $d\hat{\sigma}_{\vec{p}}(E)$ has support in $(-(3-\varepsilon)\ln|\beta|,\infty)$. Using the small β behavior of $w(\vec{p})$ and setting $k_0 = i(2m-\varepsilon)$ we find, dropping $0(\beta)$ terms, that

$$H(\vec{p}, i(2m(\beta)-\varepsilon)) = 2(2\pi)^{d-1}\frac{\langle s_1^2\rangle^2}{(1-e^{-\varepsilon})}.$$

Also $\tilde{K} = \ell$ so that

$$(\tilde{K}\tilde{D}^0(k_0)\,f)_{ij}\,(\vec{p}) = \int_{T^{d-1}} H(\vec{p},k_0)(\ell f)_{ij}\,(\vec{p})\,d\vec{p}.$$

The bound states are determined by eigenvalues of value 1 of $(2\pi)^{-2(d-1)}\,\tilde{K}\,\tilde{D}^0(k^0)$ which are found by taking $f^k(\vec{p}) = P^k u$, $k = 0, 2$, where u is a $\vec{p}$ independent vector. Substituting $f^k(\vec{p})$ in the above leads to

$$\lambda_k\int H(\vec{p},k_0)\,d\vec{p} = 1 = 2\lambda_k\langle s_1^2\rangle^2/(1-e^{-\varepsilon}),$$

i.e. the same condition for the existence of a bound state obtained previously.

Thus we have shown that for sufficiently high temperatures and in the ladder approximation that there are precisely two bound states below the two-particle threshold if $\alpha_N > 0$.

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